Soliton Solution to the (3+1)-dimensional Kadomtsev-Petviashvili Equation by the Tanh-Coth Method

Anwar Ja'afar Mohamad Jawad, Ali A. J. Adham

Abstract Exact travelling wave solutions are obtained to the (3+1)-dimensional Kadomtsev-Petviashvili equation and (2+1)-dimensional equation by means of the tanh-coth method. New solitary wave solutions are got. The method is applicable to a large variety of nonlinear partial differential equations. The outcome of this method will be powerful to solve (3+1)-dimensional, it was difficult to obtain the solution in this case.

Index Terms: Exact solution, Kadomtsev-Petviashvili equation, tanh-coth method, nonlinear partial differential equations.

1. Introduction

Soliton can be defined as a solution of a nonlinear partial differential equation. Solitons are found in many physical phenomena. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. Solitons are solitary waves with elastic scattering property. Due to dynamical balance between the nonlinear and dispersive effects these waves retain their shapes and speed to a stable waveform after colliding with each other. One basic expression of a solitary wave solution is of the form [1]:

$$u(x,t) = f(x-\lambda t) \qquad (1)$$

where λ is the speed of wave propagation. For $\lambda > 0$, the wave moves in the positive x direction, whereas the wave moves in the negative x direction for $\lambda < 0$.

Travelling waves, whether their solution expressions are in explicit or implicit forms are very interesting from the point of view of applications.

Corresponding Author:
Anwar Ja'afar Mohamad Jawad
Al-Rafidain University College , Baghdad, Iraq
anwar_jawad2001@yahoo.com

Co-Author
Ali A. J. Adham
Faculty of Technology University Malaysia Pahang,
aliadham@ump.edu.my

These types of waves will not change their shapes during propagation and are thus easy to detect. Of particular interest are three types of travelling waves: the solitary waves, which are localized travelling waves, asymptotically zero at large distances, the periodic waves, which rise or descend from one asymptotic state to another. Recently, algebraic method, called the mapping method [2-3], is proposed to obtain exact travelling wave solutions for a large variety of nonlinear partial differential equations (PDEs). This method includes several direct methods as special cases, such as tanh-function method[4], sech-function method, and tanh-coth method[5].

1.1. The Kadomtsev-Petviashvili (KP) Equation

The nonlinear dispersive equation formulated by Korteweg and de Vries (KdV) in its simplest form is (1+1) dimensional equation and is given by [6]:

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (2)$$

with $u = u(x,t)$ is a differentiable function. We shall assume that the solution $u(x,t)$, along with its derivatives, tends to zero as $|x| \to \infty$.

In 1970, Kadomtsev and Petviashvili generalized the KdV (1+1) dimensional equation to two space variables (2+1) dimensional equation and formulated the well-known Kadomtsev-Petviashvili equation to provide an explanation of the general weakly dispersive waves [7]. They developed this equation when they relaxed the restriction that the waves be strictly one-dimensional of the KdV equation.

1.2. The (2+1)-dimensional Kadomtsev-Petviashvili equation

The (2+1)-dimensional Kadomtsev-Petviashvili equation is given by:

$$(u_t - 6uu_x + u_{xxx})_x + \alpha.u_{yy} = 0 \qquad (3)$$

where $u = u(x, y, t)$ is a real-valued function of two spatial variables x and y, and one time variable t, and α is a constant scalar. When $\alpha = 0$, Eq. (3) reduces to the KdV equation Eq.(2). When $\alpha < 0$, the equation is known as the KP-I equation which is a good model when surface tension is strong and dominates in very shallow water. However, for $\alpha > 0$, the equation is called the KP-II equation which is a good model when surface tension is weak or absent. In other words, the coefficients $\alpha > 0$ and $\alpha < 0$ are used for weak surface tension and strong surface tension respectively. This means that the two KP equations have different physical structures and different properties. The KP equation is used to model shallow-water waves with weakly non-linear restoring forces. It is also used to model waves in ferromagnetic media. Several analytical and numerical approaches were employed to solve the KP equations. The KP solutions have been studied extensively to derive periodic and soliton solutions.[1]

1.3. The (3+1)-dimensional modified Korteweg-de Vries (kdV) equation

The (3+1)-dimensional modified Korteweg-de Vries (kdV) equation is given by [8]:

$$u_t + 6u^2 u_x + u_{xyz} = 0 \qquad (4)$$

and the Solitary wave solution is given by

$$u(x, y, z, t) = \pm\sqrt{c_2 c_3}\ \mathrm{sec}\,h$$
$$[c_1 x + c_2 y + c_3 z - c_1 c_2 c_3 t + \delta] \qquad (5)$$

1.4. The (3+1)-dimensional KP equation

The (3+1)-dimensional KP equation given by [9,10].

$$(u_t + 6uu_x + u_{xxx})_x + \alpha(u_{yy} + u_{zz}) = 0 \qquad (6)$$

Mohammed [11] constructed new exact traveling wave solutions of the (3+1) dimensional Kadomtsev-Petviashvili (KP) equation, in which the homogeneous balance method is applied to solve the Riccati equation and the reduced nonlinear ordinary differential equation, respectively.

Peng et al [12] obtained exact travelling wave solutions in terms of the Jacobi elliptic functions to the (3+1)-dimensional Kadomtsev-Petviashvili equation by means of the extended mapping method.

This paper aims to implement tanh-coth method to determine solitary solution for the (3+1)-dimensional KP equation.

2. Outline of Tanh-Coth Method

For a given nonlinear evolution equation, say, in four variables

$$P(u, u_t, u_x, u_y, u_z, u_{xx}.........) = 0 \qquad (7)$$

We seek a travelling wave solution of the form:

$$u(x, y, z, t) = U(\xi), \quad \text{and}$$
$$\xi = kx + \alpha y + \beta z + \omega t + \theta_0 \qquad (8)$$

Where $k, \alpha, \beta, \omega, \theta_0$ are constants. The following chain rule

$$\frac{\partial U}{\partial t} = \omega\frac{dU}{d\xi}, \quad \frac{\partial U}{\partial x} = k\frac{dU}{d\xi},$$
$$\frac{\partial U}{\partial y} = \alpha\frac{dU}{d\xi}, \quad \frac{\partial U}{\partial z} = \beta\frac{dU}{d\xi},$$
$$\frac{\partial^2 U}{\partial x^2} = k^2\frac{d^2 U}{d\xi^2}$$

converted the PDE Eq.(3), to an ordinary differential equation ODE

$$Q(U, U', U'', U''', ...) = 0 \qquad (9)$$

with Q being another polynomial form of there argument, which will be called the reduced ordinary differential equations of Eq.(9). Integrating Eq.(9) as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well. Now finding the traveling wave solutions to Eq.(9) is equivalent to obtaining the solution to the reduced ordinary differential equation Eq.(9).

The key step is to introduce the ansatz, the new independent variable

$$Y = \tanh(\xi) \qquad (10)$$

that leads to the change of variables:

$$\frac{dU}{d\xi} = (1 - Y^2)\frac{dU}{dY}$$
$$\frac{d^2U}{d\xi^2} = -2Y(1 - Y^2)\frac{dU}{dY} + (1 - Y^2)^2\frac{d^2U}{dY^2}$$
$$\frac{d^3U}{d\xi^3} = 2(1 - Y^2)(3Y^2 - 1)\frac{dU}{dY}$$
$$-6Y(1 - Y^2)^2\frac{d^2U}{dY^2} + (1 - Y^2)^3\frac{d^3U}{dY^3} \qquad (11)$$

The next step is that the solution is expressed in the form [12]

$$U(\xi)=\sum_{i=0}^{m}a_iY^i+\sum_{i=1}^{m}b_iY^{-i} \quad (12)$$

where the parameter m can be found by balancing the highest-order linear term with the nonlinear terms in Eq.(9), and $k,\alpha,\beta,\omega,a_0,a_1,....,a_m,b_1,....,b_m$ are to be determined. Substituting Eq.(12) into Eq.(9) will yield a set of algebraic equations for $k,\alpha,\beta,\omega,a_0,a_1,....,a_m,b_1,....,b_m$ because all coefficients of Y have to vanish. Having determined these parameters, knowing that m is positive integer in most cases, and using Eq.(12) we obtain analytic solutions $u(x,t)$,in a closed form.

The hyperbolic functions can be extended to trigonometric functions by using the complex form. So that a tan-function expansion solution generates from a tanh function expansion solution for $Y=\tanh(i\xi)=i\tan(\xi)$, and a cot-function expansion solution generates from a coth function expansion solution for $Y^{-1}=\coth(i\xi)=-i\cot(\xi)$. The tanh-coth method seems to be powerful tool in dealing with nonlinear physical models [13].

3. Applications

3.1. Exact solutions to the (3+1)-dimensional KP equation

The (3 + 1)-dimensional KP-I equation is given by [9,10]:

$$(u_t+6uu_x+u_{xxx})_x-3(u_{yy}+u_{zz})=0 \quad (13)$$

This explains wave propagation in the field of plasma physics, fluid dynamics, etc. Soliton simulation studies for Eq.(13) have been done by Senatorski et al. [14].

To study the travelling wave solutions to Eq.(13), substitute $u(x,y,z,t)=U(\xi)$, and $\xi=kx+\alpha y+\beta z+\omega t+\theta_0$ into Eq.(13) and integrating twice, we have:

$$k^4U^{//}+[k\omega-3(\alpha^2+\beta^2)]U+\frac{3}{2}k^2U^2=0 \quad (14)$$

we postulate tanh series , and the transformation given in Eq.(10), so that Eq.(14) reduces to:

$$k^4[-2Y(1-Y^2)\frac{dU}{dY}+(1-Y^2)^2\frac{d^2U}{dY^2}]+AU+\frac{3}{2}k^2U^2=0 \quad (15)$$

where:

$$A=[k\omega-3(\alpha^2+\beta^2)] \quad (16)$$

Now, to determine the parameter m, we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq.(15) we balance U^2 with $U^{//}$, to obtain:

m+2 = 2m , then m = 2. The tanh-coth method admits the use of the finite expansion for :

$$U=a_0+a_1Y+a_2Y^2+b_1Y^{-1}+b_2Y^{-2}$$

and

$$U^{/}=a_1+2a_2Y-b_1Y^{-2}-2b_2Y^{-3}$$

and

$$U^{//}=2a_2+2b_1Y^{-3}+6b_2Y^{-4} \quad (17)$$

Substituting $U^{/},U^{//}$ from Eq.(17) in Eq.(15), then equating the coefficient of Y^i, i= 0, 1, 2, 3, 4, -1, -2, -3, -4 leads to the following nonlinear system of algebraic equations:

$$\overset{0}{Y}:-4k^4[a_2+4b_2]+Aa_0+\frac{3}{2}k^2[a_0^2+2(a_1b_1+a_2b_2)=0$$

$$Y:-2k^4[a_1+3b_1]+Aa_1+3k^2a_0a_1=0$$

$$\overset{2}{Y}:4k^4a_2+Aa_2+3k^2(a_0a_2+a_1^2)=0$$

$$\overset{3}{Y}:2k^4a_1++3k^2a_1a_2=0$$

$$\overset{4}{Y}:\frac{3}{2}k^2a_2^2=0$$

$$\overset{-1}{Y}:10k^4b_1+Ab_1+3k^2(a_0b_1+a_1b_2)=0$$

$$\overset{-2}{Y}:28k^4b_2+Ab_2+\frac{3}{2}k^2(2a_0b_2+b_1^2)=0$$

$$\overset{-3}{Y}:-4k^4b_1+3k^2b_1b_2=0$$

$$\overset{-4}{Y}:-8k^4b_2+k^2b_2^2=0$$

Solving the nonlinear systems of equations (18) we can get $a_1=0$,

ISSN 2229-5518

$\omega = \dfrac{A + 3(\alpha^2 + \beta^2)}{k}$, and $b_1 = 0$ with the following cases:

Case 1

$a_0 = \dfrac{-4}{3}k^2$, $a_2 = -2k^2$, $b_2 = -2k^2$,

$A = 16k^3$

$$u_1 = -2k^2[\frac{2}{3} + \tanh^2(\zeta) + \coth^2(\zeta)] \quad (19)$$

Case 2

$a_0 = 4k^2$, $a_2 = -2k^2$, $b_2 = -2k^2$,

$A = -16k^3$

$$u_2 = 2k^2[2 - \tan h^2(\zeta) - \coth^2(\zeta)] \quad (20)$$

Case 3

$a_0 = \dfrac{4+\sqrt{14}}{3}k^2$, $a_2 = -2k^2$,

$b_2 = \dfrac{-k^2}{3}$ $A = -2\sqrt{14}k^3$

$$u_3 = k^2[\frac{4+\sqrt{14}}{3} - 2\tanh^2(\zeta) - \frac{1}{3}\coth^2(\zeta)] \quad (21)$$

Case 4

$a_0 = \dfrac{4-\sqrt{14}}{3}k^2$, $a_2 = -2k^2$,

$b_2 = \dfrac{-k^2}{3}$, $A = 2\sqrt{14}k^3$

$$u_4 = k^2[\frac{4-\sqrt{14}}{3} - 2\tanh^2(\zeta) - \frac{1}{3}\coth^2(\zeta)] \quad (22)$$

Case 5

$a_0 = \dfrac{4-\sqrt{14}}{3}k^2$, $a_2 = \dfrac{-k^2}{3}$,

$b_2 = -2k^2$, $A = 2\sqrt{14}k^3$

$$u_5 = k^2[\frac{4-\sqrt{14}}{3} - \frac{1}{3}\tanh^2(\zeta) - 2\coth^2(\zeta)] \quad (23)$$

Case 6

$a_0 = \dfrac{4+\sqrt{14}}{3}k^2$, $a_2 = \dfrac{-k^2}{3}$,

$b_2 = -2k^2$, $A = -2\sqrt{14}k^3$

$$u_6 = k^2[\frac{4+\sqrt{14}}{3} - \frac{1}{3}\tanh^2(\zeta) - 2\coth^2(\zeta)] \quad (24)$$

Case 7

$a_0 = \dfrac{4+\sqrt{14}}{3}k^2$, $a_2 = \dfrac{-k^2}{3}$,

$b_2 = \dfrac{-k^2}{3}$, $A = -2\sqrt{14}k^3$

$$u_7 = k^2[\frac{4+\sqrt{14}}{3} - \frac{1}{3}\tanh^2(\zeta) - \frac{1}{3}\coth^2(\zeta)] \quad (25)$$

Case 8

$a_0 = \dfrac{4-\sqrt{14}}{3}k^2$, $a_2 = \dfrac{-k^2}{3}$,

$b_2 = \dfrac{-k^2}{3}$, $A = 2\sqrt{14}k^3$

$$u_8 = k^2[\frac{4-\sqrt{14}}{3} - \frac{1}{3}\tanh^2(\zeta) - \frac{1}{3}\coth^2(\zeta)] \quad (26)$$

Results of solving (3+1) dimensional KP in this paper are compatible with that results obtained by Wazwaz [15] for the (2+1) dimensional KP equation.

For $k = \alpha = \beta = 1, \theta_0 = 0$, the solitary solution in Eq.(19) becomes:

$$u_1(x,y,z,t) = -2[\frac{2}{3} + \tanh^2(x+y+z+22t) + \coth^2(x+y+z+22t)] \quad (27)$$

and is shown in figure (1) for given t, x, y, z.

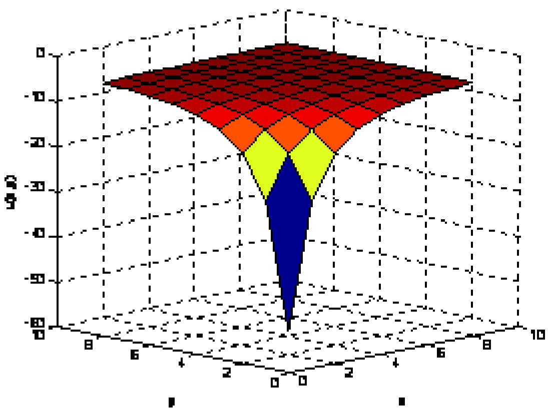

Figure (1) Solitary solution for $0.1 \le y \le 1$, $0.1 \le x \le 1$, $t = 0, z = 0$

3.2. Exact solution to the (2+1)-dimensional equation

The (2 + 1)-dimensional equation

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy} = 0 \quad (28)$$

Substituting $u(x,y,t)=U(\xi)$,and $\xi = kx+\alpha y+\omega t+\theta_0$ into Eq. (28) and integrating once with zero constant, we have:

$$k^2\alpha u^{///}+\omega u^{/}-3k\alpha u^{/2}=0 \qquad (29)$$

we postulate the following tanh series , and the transformation given in Eq.(4), then Eq.(29) reduces to:

$$k^2\alpha[2(1-Y^2)(3Y^2-1)\frac{dU}{dY}-6Y(1-Y^2)^2\frac{d^2U}{dY^2}$$
$$+(1-Y^2)^3\frac{d^3U}{dY^3}]+\omega[(1-Y^2)\frac{dU}{dY}]-3k\alpha[(1-Y^2)\frac{dU}{dY}]^2=0 \qquad (30)$$

Now, to determine the parameter m, we balance the linear term of highest-order with the highest order nonlinear terms. So, in Eq. (30) we balance $U^{/2}$ with $U^{///}$, to obtain m+3 = 2m+2 , then m=1. The tanh-coth method admits the use of the finite expansion for :

$$U=a_0+a_1Y+b_1Y^{-1}$$

and $U^{/}=a_1-b_1Y^{-2}$, $U^{//}=2b_1Y^{-3}$, and $U^{///}=-6b_1Y^{-4}$ (31)

Substituting $U^{/},U^{//},U^{///}$ from Eq.(31) in Eq. (30), then equating the coefficient of Y^i , i= 0, 2, 4, -2,-4 leads to the following nonlinear system of algebraic equations:

$$Y^0:-2k^2\alpha(a_1+b_1)+\omega(a_1+b_1)$$
$$-3k\alpha(a_1+b_1)^2-6k\alpha a_1b_1=0$$
$$Y^2:8k^2\alpha a_1-a_1\omega+6k\alpha(a_1+b_1)a_1=0 \qquad (32)$$
$$Y^4:-6k^2\alpha a_1-3k\alpha a_1^2=0$$
$$Y^{-2}:8k^2\alpha b_1+\omega b_1+6k\alpha(a_1+b_1)b_1=0$$
$$Y^{-4}:-6k^2\alpha b_1-3k\alpha b_1^2=0$$

Solving the nonlinear systems of equations (32) we can get:

$a_1=-2k$, $\omega=-16k^2\alpha$, $b_1=-2k$

$$u=a_0-2k\ [\tanh(kx+\alpha y-16k^2\alpha t+\theta_0)$$
$$+\coth(kx+\alpha y-16k^2\alpha t+\theta_0)] \qquad (33)$$

Or

$$u=a_0-4k\coth[2(kx+\alpha y-16k^2\alpha t+\theta_0)] \qquad (34)$$

Remark: the solitary solution (34) is verifying the problem equation in (28).

For $a_0=1, k=\alpha=0.5, \theta_0=0$ the solitary solution in Eq.(33) is

$$u(x,y,t)=1-2\coth[x+y-4t]$$

and is shown in figure (2).

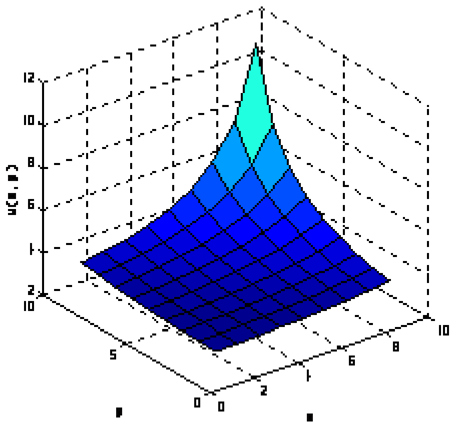

Figure (2) solitary solution u(x,y) for $t=0.5$, $0.1\le x\le 1, 0.1\le y\le 1$.

4. Conclusion

The exact travelling wave solutions to (3+1)-dimensional KP and (2+1)- dimensional equations have been studied by means of tanh-coth method. It can be easily seen that the implemented methods used in this paper are powerful and applicable to a large variety of nonlinear partial differential equations.

References

[1] Wazwaz A. M. , Partial Differential Equations and Solitary Waves Theory, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg , 2009.
[2] Peng Y.-Z. Acta Phys. Pol. A 103, 417 , 2003.
[3] E. Yomba, Chaos, Solitons and Fractals 21, 209 , 2004.
[4] Jawad A. J. M , Petkovi'c M. D. , Biswas A. , Soliton solutions of a few nonlinear wave equations, Applied Mathematics and Computation journal, Vol. 216 , pp. 2649–2658 , 2010.
[5] Parkes E. J. , Observations on the tanh-coth expansion method for finding solutions to nonlinear evolution equations, Appl. Math. Comput. Vol. 217 , pp. 1749–1754, 2010.
[6] Drazin P, G., and Johnson R.S., Solitons: an Introduction, Cambridge University Press, Cambridge, (1996).
[7] Kadomtsev B. B. and Petviashvili V. I., On the stability of solitary waves in weakly dispersive media, Sov. Phys. Dokl., 15, 539–541, 1970.
[8] Willy Hereman, Exact Solutions of Nonlinear Partial Differential Equations The Tanh/Sech Method, Wolfram Research Inc., Champaign, Illinois October 25– November 11, 2000.
[9] Ablowitz M., Segur H., J. Fluid Mech. 92, 69, 1979.
[10] Kuznietsov E.A. , Musher C.L., Exp J.. Theor. Phys. 63, 947 , 1986.
[11] Mohammed K., New Exact traveling wave solutions of the (3+1) dimensional Kadomtsev-Petviashvili (KP) equation, International Journal of Applied Mathematics, vol. 37 : 1 ,2007.

[12] PengamY. Z., and Krishnanb E. V., Exact Travelling Wave Solutions to the (3+1)-Dimensional Kadomtsev-Petviashvili Equation, Acta physica Polonica A, Vol. 108 ,2005.
[13] Abdou M. A. , The extended tanh method and its applications for solving nonlinear physical models, Applied Mathematics and Computation journal, Vol. 190 , 988–996,2007.
[14] Senatorski A., Infed E., Phys. Rev. Lett. 77, 2855 (1986).
[15] A. M.Wazwaz, Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh–coth method, Applied Mathematics and Computation 190 , 633–640, 2007.

The Author: **Dr. Anwar Ja'afar Mohamed Jawad** was born in Baghdad, Iraq (1961). He is one of the academic staff in Al-Rafidain University College, Baghdad-Iraq. He is an Assistant professor in Applied Mathematics. The academic Qualifications are PhD. in Applied Mathematics from University of Technology, Baghdad, (2000), M.Sc. in Operation Research from University of Technology, Baghdad, (1989), and B.Sc. in Mechanical Engineering from Baghdad University, (1983). He is interested in Differential equations, and Numerical analysis. He published in international journals more than 40 manuscripts in solving nonlinear partial differential equations. He was teaching Mathematics, numerical analysis for graduate and postgraduate students in Iraqi and Syrian universities. He was a supervised for many MSc. and PhD. thesis.

Co-Author: **Dr. Ali A. J Adham** : In 1995, was obtained his B.SC from Mosul University/Iraq, Faculty of Computer Science and Mathematics. In 2005, he obtained the M.SC degree from University of Technology/Iraq, Faculty of Applied Science. The PhD degree obtained from University Malaysia Pahang/ Malaysia in 2012. His interest is in Applied Science and Operation Research. He has teaching experience more than 16 years in several universities and Colleges in teaching many subjects under computer science, operation research, and Mathematic.