Heat conduction problem on semi-infinite solid cylinder with heat source

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Abstract— This article concerned with the determination of temperature and thermal stresses of semi-infinite solid cylinder with cylindrical surface heat source having constant initial temperature under unsteady-state field. The lower boundary surface is kept insulated and upper surface of the cylinder is at initial temperature while heat is dissipated by convection from boundary surface at r=b in to a medium at zero temperature .The result are obtained in series form in term of Bessel's functions these have been computed numerically and graphically.

Keywords- Non homogeneous heat condition, heat generation, semi-infinite hollow cylinder, integral Transform

1 INTRODUCTION

THE study of thermal effect on deformation and stresses of cylinder especially solid cylinder is increasingly important. The problem of semi infinite cylinder is more complicated and thus more attractive to many scientists in various modern projects, such as high building, raceway, and container and so on. Astumi et al [9] has discussed the linear thermoelastic problem of infinitely long circular cylinder with a circumference edge crack thermal stresses cause by uniform heat flow distributed by the presence of the crack. The crack surface and cylindrical surfaces are insulated. Noda et al [7] has explained the two dimension problem of infinite long circular cylinder whose lateral surface is traction free and subjected to an asymmetrical heating is consider within the context of the theory of generalized thermo elasticity with one relaxation time. Dos-Sung Lee [3]discussed the three dimensional analysis of the stress distribution in long circular cylinder containing a concentric very thin spherical cap cavity .the central plane of cavity is perpendicular to the axis of the cylinder and the cylinder is subjected to bending .also Kulkarni et al [1]determine the quasi –static thermal stresses in thick circular plate subjected to arbitrary initial temperature on the upper surface with lower surface at zero temperature and fixed circular edge thermally insulated

In this work we study the thermoelastic problem of semi-infinite solid cylinder with cylindrical surface heat source having constant initial temperature under unsteady-state field. To determine the temperature and thermal stresses by using Hankel and Fourier transform technique.

2. FORMULATION OF THE PROBLEM

Consider a semi-infinite solid cylinder defined $0 \le r \le b$, $0 \le z < \infty$, by having constant initial temperature T_0

Let (r, φ, z) be cylindrical co-ordinate system and θ be temperature function of space and time the transient heat conduction equation with internal heat generation is given as

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \frac{\xi(r\,,z\,,y,\theta)}{K} = \frac{1}{K}\frac{\partial T}{\partial t} \tag{2.1}$$

Where $\xi(r\,,z\,,t,\theta)$ is the internal heat source function, and the $\kappa = \frac{\lambda}{\rho C}$,λ being the thermal conductivity of the material, ρ is density and C is the capacity

Following [11] use the substitution

$$\xi(r\,,z\,,t,\theta) = \Phi(r,z,t) + \psi(t)\theta(r,z,t) \tag{2.2}$$

and

$$T(r\,,z\,,t) = \theta(r,z,t)e^{-\int_0^t \psi(y)dy} \tag{2.3}$$

$$\chi(r\,,z\,,t) = \Phi(r,z,t)e^{-\int_0^t \psi(y)dy} \tag{2.4}$$

For the sake of brevity, we consider

$$\chi(r\,,z\,,t) = \frac{\delta(r-r_0)\delta(z)}{2\pi r_0}e^{-\omega t}\ ,\quad 0 \le r_0 \le b, 0 \le z < \infty, \omega > 0 \tag{2.5}$$

Substitute the equation (2.2) to (2.5) in (2.1), we obtain

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{\chi(r\,,z\,,t))}{K} = \frac{1}{K}\frac{\partial T}{\partial t} \tag{2.6}$$

where K is the thermal diffusivity of the material of the cylinder, subjected to the initial condition and boundary conditions.

$$T = T_0 \ \text{ at } t = 0 \tag{2.7}$$

$$T + k_1\frac{\partial T}{\partial r} = 0 \ \text{ at } r=b\ , 0 \le z < \infty, t>0 \tag{2.8}$$

$$\frac{\partial T}{\partial z} = 0 \ \text{ at } z=0\ , 0 \le r < b, t>0 \tag{2.9}$$

$$T = T_0 \ \text{ at } z \to \infty\ \ , 0 \le r < b \tag{2.10}$$

Where $T = T(r,z,t)$

The displacement function in the cylindrical co-ordinate system are represented by the Goodier's thermoelastic displacement potential and Love's function as [4]

$$u_r = \frac{\partial\phi}{\partial r} - \frac{\partial^2\phi}{\partial r\,\partial z} \tag{2.11}$$

$$u_z = \frac{\partial\phi}{\partial z} + 2(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \tag{2.12}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2\phi = \left(\frac{1+\nu}{1-\nu}\right) a_t\theta \tag{2.13}$$
$$\nabla^2(\nabla^2 L) = 0 \tag{2.14}$$
Where $\nabla^2 = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}$

The component of the stresses are represented by the use of the potential ϕ and Love's L function as
$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2\phi}{\partial r^2} - \nabla^2\phi\right) + \frac{\partial}{\partial z}\left(\upsilon\nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right)\right\} \tag{2.15}$$
$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial r} - \nabla^2\phi\right) + \frac{\partial}{\partial z}\left(\upsilon\nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right)\right\} \tag{2.16}$$
$$\sigma_{zz} = 2G\left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial r} - \nabla^2\phi\right) + \frac{\partial}{\partial z}\left(\upsilon\nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right)\right\} \tag{2.17}$$
$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\,\partial z} + \frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\} \tag{2.18}$$
Where G and υ are the shear modulus and Poisson's ratio respectively the boundary condition on the traction free surface functions are $\sigma_{rr} = \sigma_{rr} = 0$ at $r = b$ (2.19)

Equation (2.1) to (2.19) constitutes the mathematical formation of the problem under consideration.

3. Solution of the Problem

We introduce Hankel Transform and its inverse [10]
$$\bar{F}(\mu_m) = \int_{r'=0}^{b} r' k_0(\mu_m, r')\, F(r')\,dr'$$
$$F(r) = \sum_{m=1}^{\infty} k_0(\mu_m, r)\, \bar{F}(\mu_m)$$
Where $k_0(\mu_m, r) = \frac{\sqrt{2}}{b}\frac{\mu_m}{(k_1{}^2+\mu_m{}^2)^{1/2}}\frac{J_0(\mu_m r)}{J_0(\mu_b)}$, and Eigen value μ_m are positive root of $\mu J'_0(\mu_b) + k_1 J_0(\mu_b) = 0$

Applying Hankel Transform to (2.3),(2.4),(2.5) , (2.7)and using (2.6) we obtain
$$K\left[-\mu_m{}^2\bar{T}(m,z,t) + \varphi + \frac{\partial^2\bar{T}(m,z,t)}{\partial z^2}\right] + \bar{\chi}(m,z,t) = \frac{\partial\bar{T}(m,z,t)}{\partial z} \tag{3.1}$$
$$\bar{T} = \bar{T}_0,\ \frac{\partial\bar{T}}{\partial r} = 0 \tag{3.2}$$
$$\bar{\chi}(m,z,t) = \frac{\sqrt{2}}{b}\frac{\mu_m}{(k_1{}^2+\mu_m{}^2)^{1/2}}\frac{J_0(\mu_m r_0)}{J_0(\mu_b)}\delta(z)e^{-\omega t} \tag{3.3}$$
We introduce Fourier Transform and it inverse for region $0 \le z < \infty$ [10]
$$\bar{F}(\beta) = \int_{z'=0}^{\infty} k(\beta, z')\, F(z')\,dz'$$
$$F(z) = \int_{\beta=0}^{\infty} k(\beta, z)\, \bar{F}(\beta)\,d\beta$$
Where $k(\beta, z) = \sqrt{\frac{2}{\pi}}\cos\beta z$, Eigen value β is positive root of $\cos\beta z = 0$
$$K[-\mu_m{}^2\bar{T}^*(m,\beta,t) - \beta^2\bar{T}^*(m,\beta,t)] + \bar{\chi}^*(m,\beta,t) = \frac{d\bar{T}^*(m,\beta,t)}{dz} \tag{3.4}$$
$$\frac{d\bar{T}^*(m,\beta,t)}{dz} + K(\mu_m{}^2 + \beta^2)\bar{T}^*(m,\beta,t) = H(\mu_m,\beta) \tag{3.5}$$
Where
$$H(\mu_m,\beta) = \bar{\chi}^*(m,\beta,t) = \frac{\sqrt{2}}{b}\frac{\mu_m}{(k_1{}^2+\mu_m{}^2)^{1/2}}\frac{J_0(\mu_m r_0)}{J_0(\mu_b)}\int_{\beta=0}^{\infty}\sqrt{\frac{2}{\pi}}\cos\beta z_0\,d\beta\, e^{-\omega t}$$
which is the first order differential equation and has solution
$$\bar{T}^*(m,\beta,t)e^{K(\mu_m{}^2+\beta^2)t} = \frac{H(\mu_m,\beta)}{K(\mu_m{}^2+\beta^2)-\omega}e^{-K(\mu_m{}^2+\beta^2)t} + C \tag{3.6}$$
At initially t=0
$$C = \bar{T}^*{}_0 - \frac{H(\mu_m,\beta)}{K(\mu_m{}^2+\beta^2)-\omega}$$
$$\bar{T}^*(m,\beta,t) = \frac{H(\mu_m,\beta)}{K(\mu_m{}^2+\beta^2)-\omega} + \left[\bar{T}^*{}_0 - \frac{H(\mu_m,\beta)}{K(\mu_m{}^2+\beta^2)-\omega}\right]e^{-K(\mu_m{}^2+\beta^2)t} \tag{3.7}$$
Applying inverse of Hankel and Fourier Transform we obtain
$$T(r,z,t) = \frac{\sqrt{2}}{b}\sum_{m=1}^{\infty}\frac{\mu_m}{(k_1{}^2+\mu_m{}^2)^{1/2}}\frac{J_0(\mu_m r)}{J_0(\mu_b)}\int_{\beta=0}^{\infty}\sqrt{\frac{2}{\pi}}\cos\beta z\,d\beta\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\} \tag{3.8}$$
Where $\Lambda_{m,n} = \frac{H(\mu_m,\beta)}{K(\mu_m{}^2+\beta^2)-\omega}$

Substitute the value of $T(r,z,t)$ in (2.3), the Temperature distribution is finally represented by
$$\theta(r,z,t) = \frac{\sqrt{2}}{b}\sum_{m=1}^{\infty}\frac{\mu_m}{(k_1{}^2+\mu_m{}^2)^{1/2}}\frac{J_0(\mu_m r)}{J_0(\mu_b)}\int_{\beta=0}^{\infty}\sqrt{\frac{2}{\pi}}\cos\beta z\,d\beta\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}\ e^{\int_0^t \psi(y)dy} \tag{3.9}$$

4. Determination of Thermoelastic Displacement

Substitute the value of $\theta(r,z,t)$in (2.13)
$$\phi = \sum_{m=1}^{\infty}\aleph\, J_0(\mu_m r)\,\varpi\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}\ e^{\int_0^t \psi(y)dy} \tag{4.1}$$
Where $\aleph = \left(\frac{1+\nu}{1-\nu}\right)a_t\frac{\sqrt{2}}{b}\frac{-1}{(\mu_m{}^2+\beta^2)}\frac{\mu_m}{J_0(\mu_b)(k_1{}^2+\mu_m{}^2)^{\frac{1}{2}}}$,
$$\varpi = \int_{\beta=0}^{\infty}\sqrt{\frac{2}{\pi}}\cos\beta z\,d\beta$$
Similarly , for solution for Love's function
$$L = \sum_{m=1}^{\infty}\aleph\frac{J_0(\mu_m r)}{J_0(\mu_b)}(z\cos\beta z)\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}\ e^{\int_0^t \psi(y)dy} \tag{4.2}$$
Substitute the value of ϕ and L in (2.11) ,(2.12) resp.
$$u_r = \sum_{m=1}^{\infty}\aleph\mu_m\, J'_0(\mu_m r)\,\varpi(1 - z\beta\cos\beta z - \sin\beta z)\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}\ e^{\int_0^t \psi(y)dy} \tag{3.3}$$
$$u_z = \sum_{m=1}^{\infty}\aleph\, J_0(\mu_m r)\,\varpi\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}[\varpi' - 2(1-\nu)\mu_m{}^2 - z\beta\sin\beta z - \cos\beta z]\ e^{\int_0^t \psi(y)dy} \tag{4.4}$$

5. Determination of Stress Functions

Using (2.13),(2.14),(4.3) and (4.4) in (2.15),(2.16),(2.17)and (2.18) the stress functions are obtained as
$$\sigma_{rr} = 2G\sum_{m=1}^{\infty}\aleph\, J_0(\mu_m r)\,\varpi\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}\{\varpi\left[\frac{-1}{(\mu_m{}^2+\beta^2)}J''_0(\mu_m r) + J_0(\mu_m r)\right] - \frac{1}{(\mu_m{}^2+\beta^2)}[2\upsilon\beta^2\cos\beta z\, J_0(\mu_m r)] - [(\beta+1)\cos\beta z + \beta\sin\beta z]J''_0(\mu_m r)\}e^{\int_0^t \psi(y)dy} \tag{5.1}$$
$$\sigma_{\theta\theta} = 2G\sum_{m=1}^{\infty}\aleph\, J_0(\mu_m r)\,\varpi\left\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\right\}\{\varpi\left[\frac{-1r}{(\mu_m{}^2+\beta^2)}J'_0(\mu_m r) + J_0(\mu_m r)\right] - \frac{1}{(\mu_m{}^2+\beta^2)}[2\upsilon\beta^2\cos\beta z\, J_0(\mu_m r)] - [(\beta+1)\cos\beta z + \beta\sin\beta z]r^{-1}J'_0(\mu_m r)\}e^{\int_0^t \psi(y)dy}$$

(5.2)

$$\sigma_{zz=} 2G\sum_{m=1}^{\infty} \aleph\, J_0(\mu_m r)\, \varpi \Big\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\Big\}\Big\{\varpi\Big[\frac{-1}{(\mu_m{}^2+\beta^2)} J''_0(\mu_m r) + J_0(\mu_m r)\Big] + [(2\upsilon+\beta)\beta^2\cos\beta z + \beta^3 z\sin\beta z] J_0(\mu_m r)\Big]\, e^{\int_0^t \psi(y)dy} \quad (5.3)$$

$$\sigma_{rz=} 2G\sum_{m=1}^{\infty} \aleph\, J_0(\mu_m r)\, \varpi \Big\{\Lambda_{m,n} + [\bar{T}^*{}_0 - \Lambda_{m,n}]e^{-K(\mu_m{}^2+\beta^2)t}\Big\}\{\varpi'\, J_0(\mu_m r) + 2\upsilon\beta\sin\beta z + \beta^2\sin\beta z\, J'_0(\mu_m r)\ e^{\int_0^t \psi(y)dy} \quad (5.4)$$

6. Special Case and Numerical Calculation

Setting,

$$\psi(y) = -y\,, T_{0=0}\ \chi(r\,,z\,,t) = \frac{\delta(r-r_0)\delta(z)}{2\pi r_0} e^{-\omega t} \qquad (6.1)$$

$$\int_0^t \psi(y)dy = \frac{-t^2}{2}\,, \bar{T}^*{}_0 = 0 \qquad (6.2)$$

Substitute the value of equation (6.1)and (6.2)in(3.9),(4.3),(4.4)(5.1)to(5.4)we obtain the expression for the temperature and stresses respectively

$$\theta(r,z,t) = \frac{4}{b^2(4\pi kt)^{1/2}}\sum_{m=1}^{\infty}\frac{e^{-K(\mu_m{}^2)t}k_1{}^2}{(k_1{}^2+\mu_m{}^2)}\frac{J_0(\mu_m r)J_0(\mu_m r_0)}{J_1{}^2(\mu_m b)}e^{-z/4kt}\Big(1 - e^{-K(\mu_m{}^2+\beta^2)t}\Big)e^{-t^2/2} \qquad (6.3)$$

Numerical calculation have been carried out for a steel (SN 50C) cylinder with parameter b=1 $k_1 = 0.5$, z=1, t=1 thermal diffusivity k=15.9, Poisson ratio $v = 0281$ modulus of elasticity $E = 6.9 \times 10^6$, shear modulus $G = 2.7 \times 10^6$ with $\mu_m = 1.0128, 1.01234, 3.8653,\ 3.86531,$ 3.5351, 7.0257, 7.02573, 7.02572, 13.3235, 10.7183 which are the positive root of $\mu_m J'_0(br) + k_1 J_0(\mu_m b) = 0$ and $\beta = 1.5708$, 1.5708,-4.7123, 4.7123, 4.7123, 1.5708, 7.2539, 7.2539, 7.2539, 10.9951 are the positive transcendental equation of $\cos\beta z = 0$

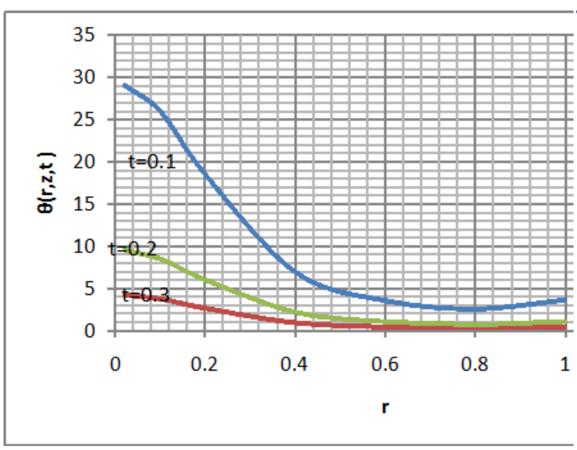

Fig.1 Temperature distribution verses radius r with different time

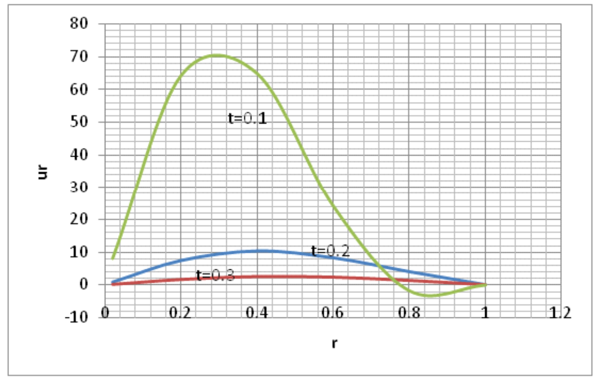

Fig.2 Radical displacement verses radius r with different time

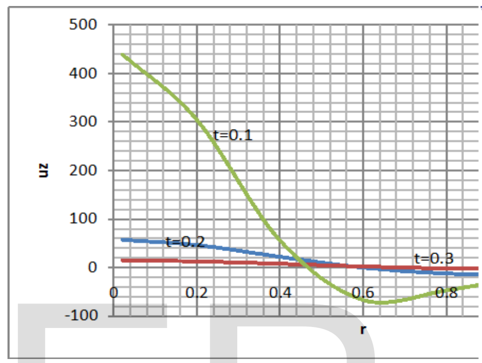

Fig.3 Tangential displacement verses radius r with different time

7. Conclusion

In this paper, we have discussed thermoelastic problem of semi-infinite solid cylinder in which heat source to the cylindrical surface of cylinder and obtained the temperature distribution and stresses we analyze particular case with mathematical model for and numerical calculations were carried out by using Hankel and Fourier transform technique we may calculate that the system of equation in this study can use to design of useful structure of machines in engineering application. Any particular case of special interest can be derived by assigning suitable value of the parameters and function in the expression.

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